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The number of colorings of planar graphs with no separating triangles.

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Abstract

A classical result of Birkhoff and Lewis implies that every planar graph with n vertices has at least $15 \cdot 2^{n-1}$ distinct 5-vertex-colorings. Equality holds for planar triangulations with $n-4$ separating triangles. We show that, if a planar graph has no separating triangle, then it has at least $(2+10^{-12})^n$ distinct 5-vertex-colorings. A similar result holds for k-colorings for each fixed $k \geq 5$. Infinitely many planar graphs without separating triangles have less than 2.252^n distinct 5-vertex-colorings. As an auxiliary result we provide a complete description of the infinite 6-regular planar triangulations.

Keywords: chromatic polynomial, planar triangulations
MSC(2000): 05C10, 05C20

1 Introduction

If G is a graph and k is a nonnegative integer, then a *k-coloring* of G is a vertex-coloring where the colors are taken from the set $\{1, 2, \ldots, k\}$, and neighboring vertices get different colors. The *chromatic polynomial* $P(G, k)$ is the number of k-colorings of G. (In this definition, permuting the colors

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gives rise to a new coloring. Not all colors in $\{1, 2, \ldots, k\}$ are necessarily used.) The chromatic polynomial was introduced by Birkhoff [3] in 1912.

An old (still unsettled) conjecture by Birkhoff and Lewis [4] says that a planar graph has no chromatic root greater than or equal to 4. In fact, they proposed the following stronger conjecture.

Conjecture 1 *Let G be a planar graph on n vertices. Then, for each real number $k \geq 4$, $P(G, k) \geq k(k-1)(k-2)(k-3)^{n-3}$.*

If true, this would be best possible, as demonstrated by any planar triangulation obtained from K_4 by adding successively vertices of degree 3. Such a triangulation has $n-4$ separating triangles. And, when a graph G has a separating triangle, then it can be written as the union of two graphs G_1, G_2 having precisely that triangle in common. In this case the chromatic polynomial factorizes, namely

$$P(G, k) = P(G_1, k)P(G_2, k)/k(k-1)(k-2).$$

Therefore, planar graphs with no separating triangles are particularly interesting in this context. And one might hope that a method which improves Conjecture 1 for $k \geq 5$ might also be applicable for the interval from 4 to 5. In contrast to this, Royle [6] has shown that planar graphs have chromatic real root arbitrarily close to 4.

Birkhoff and Lewis verified Conjecture 1 for all real numbers ≥ 5. In particular, every planar graph with n vertices has at least $15 \cdot 2^{n-1}$ distinct 5-vertex-colorings. We show that, if it has no separating triangle, then it has at least $(2+10^{-12})^n$ distinct 5-vertex-colorings. There is a similar strengthening of Conjecture 1 for each natural number > 5, and hence for each real number > 5, as the chromatic polynomial increases for $k > 5$. Infinitely many planar graphs with no separating triangles have less than 2.252^n distinct 5-vertex-colorings.

The ideas in the proof are the following. We first investigate planar triangulations with no separating triangles and with many vertices of degree < 6. We show that such graphs have many 5-colorings. Then we turn to triangulations with few vertices of degree < 6. Euler's formula implies that such a graph has a vertex v such that all vertices of distance at most 100, say, from v induce a subgraph H where all vertices have degree precisely 6. We show that v can be chosen such that H can be extended to an infinite

6-regular planar triangulation. We characterize completely the infinite 6-regular triangulations. Thus we know the structure of H and we can apply that to improve Birkhoff and Lewis' lower bound on 5-colorings (in fact on k-colorings when $k \geq 5$) provided there are no separating triangles.

The terminology and notation are essentially the same as [1], [2]. A *graph* has no loops or multiple edges. A *multigraph* may have multiple edges but no loops. If G is a multigraph, and S is a set of vertices, then the subgraph $G(S)$ induced by S has vertex set S and contains precisely those edges in G which join two vertices of S. If v is a vertex in a graph G, then the *degree* of v is denoted $d(v, G)$ (or just $d(v)$). If all vertex degrees equal r, then G is *r-regular.*

If e in an edge in a graph G, then G/e denotes the graph obtained from G by contracting e.

The complete bipartite graph $K_{2,k}$ is also called a *k-rail* between the two vertices of degree k.

A *near-triangulation* is a 2-connected planar graph such that each face, except possibly one, is bounded by a triangle.

We also define an *infinite planar triangulation* as an infinite, connected planar graph where each vertex has finite degree and, for each vertex v of degree r, the graph has a cycle C of length r such that either v is the only vertex in the interior of C or v is the only vertex in the exterior of C. In other words, v is contained in r facial triangles. Note that, if the triangulation is infinite, then there may be points in the plane which are not in the graph and which are not in a face bounded by a triangle. (The cartesian product of a cycle and two-way infinite path can be drawn in the plane such that the origin is the only point in the plane which is not in the graph and which is not inside a face-boundary. That quadrangulation can be extended to a triangulation with the same property.)

A graph is finite unless it is explicitely stated that it is infinite.

2 The chromatic polynomial

In this section we review some properties of the chromatic polynomial and motivate the results of this paper. If e is an edge in a graph G, then clearly

$$P(G - e, k) = P(G, k) + P(G/e, k). \tag{1}$$

Rearranging gives the so-called *deletion/contraction formula*

$$P(G,k) = P(G-e,k) - P(G/e,k). \tag{2}$$

So, if we wish to obtain a lower bound (for $k \geq 5$) for the chromatic polynomial of a planar graph, it suffices to consider maximal planar graphs, that is, planar triangulations (because the chromatic polynomial for a planar graph is positive for $k \geq 5$ as is well-known and as we also prove below). As noted earlier, the chromatic polynomial factorizes if the triangulation has a separating triangle. So the planar triangulations with no separating triangle become particularly interesting.

There are other recursion formulae. For example, if v is a vertex of degree d in a graph G, then we have the following

$$P(G,k) = (k-d)P(G-v,k) + \sum_{i=1}^{d}(i-1)P_i(k). \tag{3}$$

where $P_i(k)$ is the sum of all chromatic polynomials obtained from G by contracting i edges incident with v. The first variant of this is perhaps due to Oxley [7]. In [10] a short proof of this formula is presented. That was motivated by a result on graph families closed under minors (that is, edge-deletions and edge-contractions) given explicitely by Woodall [11] and, independently in [10], and implicitely by Oxley [7]. For example, every toroidal graph G contains a vertex of degree at most 6. So, the recursion formula implies immediately that $P(G,k)$ is positive for each real number $k > 6$. The same is true for all (non-zero) derivatives of the chromatic polynomial $P(G,k)$. The inequality $k > 6$ cannot be strengthened because K_7 can be drawn on the torus, and $P(K_7, 6) = 0$.

Consider now a vertex v of degree 5 in a planar triangulation G with n vertices, and let v_1, v_2, v_3, v_4, v_5 be the neighbors of v in cyclic order. Let Q_i denote the chromatic polynomial of the triangulation obtained by contracting the edge vv_i. Let $P_{i,j}$ denote the chromatic polynomial of the triangulation obtained by contracting the two edges vv_i, vv_j. Applying the equation (1) twice we obtain

$$P(G-v,k) = P(G-v+v_1v_3,k) + P_{1,3}(k) = Q_1(k) + P_{1,3}(k) + P_{1,4}(k). \tag{4}$$

Inserting this in equation (3) yields the following recursion formula which is also contained in Birkhoff and Lewis [4]:

$$P(G,k) = (k-5)Q_1(k) + (k-4)P_{1,3}(k) + (k-4)P_{1,4}(k) + P_{2,4}(k) + P_{2,5}(k) + P_{3,5}(k). \tag{5}$$

There is a similar formula if v has degree 4 (with neighbors v_1, v_2, v_3, v_4 in cyclic order), namely the following

$$P(G,k) = (k-4)Q_1(k) + (k-3)P_{1,3}(k) + P_{2,4}(k). \tag{6}$$

If v has degree 3, then clearly

$$P(G,k) = (k-3)P(G-v,k). \tag{7}$$

From these equations it follows immediately by induction on n that

$$P(G,5) \geq 60 \cdot 2^{n-3}. \tag{8}$$

which is equivalent of saying that any precoloring of the outer triangle can be extended to at least 2^{n-3} distinct 5-colorings (because a triangle has 60 distinct 5-colorings). For if there is a separating triangle we apply induction to its exterior and then its interior. And if there is no separating triangle we apply equations $(5), (6)$. (All polynomials on the right hand sides of $(5), (6)$ are non-zero because they are chromatic polynomials of graphs with no loops.)

Infinitely many examples show that the inequality (8) is sharp. However, if G has no separating triangles we can improve the inequality (8), since none of the polynomials in equations $(5), (6)$ are the zero polynomial. More precisely,

$$P(G,5) \geq 5 \cdot 60 \cdot 2^{n-2-3} = (5/4) \cdot 60 \cdot 2^{n-3}. \tag{9}$$

In this inequality the factor $5/4$ is important. We say that we *gain the factor* $5/4$ in the inequality (9), and we shall use this gain to improve the inequality (8) for those planar graphs which have no separating triangles.

In the formulas $(1) - (9)$ we count colorings. In the formulas $(3) - (6)$ we delete a vertex v and we identify other vertices. These formulas remain valid if we count only those colorings where a set of vertices distinct from v are precolored. However, when we delete v and contract some edges we may identify two precolored vertices with distinct colors, and we may create an edge between two precolored vertices with the same color. The polynomials on the right hand side which count the corresponding colorings are then the zero-polynomials.

3 Planar triangulations with no separating triangles and many vertices of degree < 6

Lemma 1 *Let m be a natural number, and let F be a forest with at least m edges. Then F has a subforest F' with at least $m/2$ edges and with no path of length 3. (In other words, each component of F' is a star.)*

Proof of Lemma 1. It suffices to consider the case where F is a tree. Let v be a vertex in F. Let F_1 be the forest consisting of all edges uw where u has odd distance, say distance i, to v, and w has distance $i+1$ to v. Let F_2 be the forest consisting of all remaining edges. Then either F_1 or F_2 can play the role of F'.

The purpose of Theorem 1 below is to show that a triangulation with many vertices of degree < 6 has many vertices of degree < 6 which are centers of wheels which are nearly disjoint. These turn out to be useful for finding many colorings using the recursion formulas (5), (6).

Theorem 1 *Let G be a planar triangulation with n vertices, and let q be a natural number. Assume that G contains a set of $4000q$ vertices of degree < 6 such that no 13 of them are part of a 13-rail. Assume that G has no separating triangle. Then G contains a subgraph H with q blocks such that*

(i) each block of H is a wheel whose center is a vertex of degree < 6, and

(ii) each connected component of H has at most one cutvertex and is an induced subgraph in G.

(In other words, G may contain edges that join distinct components of H, but G has no edge e which joins two vertices belonging to the same component of H unless e is an edge of H.)

Proof of Theorem 1. Before we argue formally we make a remark on the formulation of the theorem and its proof. It is important that the graph H we end up with is an induced subgraph. In the proof below we first find a subgraph satisfying the conditions of Theorem 1 except that it may not be induced. Then we take an appropriate subgraph which is induced. In this step it is convenient that the subgraph has at most one cutvertex in each component. (It would be even better if H has no cutvertex at all but examples show that that cannot be achieved.)

We now argue formally. By the 4-color theorem, G contains a set A of $1000q$ of the $4000q$ vertices of degree < 6 no two of which are neighbors. Each vertex v in A is a center of a wheel W_v. Now let H_0 be a subgraph (not necessarily an induced subgraph) of G such that each block of H_0 is a wheel of the form W_v where v is in A. Assume H_0 is maximal with this property. Let p be the number of blocks in H_0. If u is a vertex of A not in H_0, then W_u has at least two vertices in common with H_0 because of the maximality of H_0. So W_u contains a path P_u joining two distinct vertices of H_0 such that the mid-vertex of P_u is u. There are at least $1000q - p$ such paths P_u. At most 12 of the paths P_u join the same two vertices of H_0. Now H_0 has at most $5p$ vertices and at most $5p$ edges on the outer face boundary. The number of edges that can be added to H_0 (in its outer face) preserving planarity is at most $3(5p) - 6 - 5p$. Hence there are at most $12(15p - 6 - 5p)$ paths of the form P_u joining two nonadjacent vertices in H_0, and there are at most $5p$ such paths joining two adjacent vertices in H_0 (because G has no separating triangles).

Hence

$$1000q - p \leq 12(15p - 6 - 5p) + 5p$$

and hence $p \geq 8q$.

Assume now that H_0 is connected. (Otherwise we consider each component of H_0 separately.) If H_0 has only one block, we put $H = H_0$. Otherwise, let v_0 be a cutvertex of H_0. All wheel centers in H_0 have odd distance (in H_0) to v_0. Now we use the idea in Lemma 1. More precisely, let A_1 consist of those wheel centers in H_0 which have distance 1 modulo 4 to v_0, and let A_2 consist of those wheel centers in H_0 which have distance 3 modulo 4 to v_0. (Note that distances in H_0 are not the same as the distances in G. For example, there may be edges between the cutvertices of H_0.) We wish to end up with an induced subgraph. First we shall find an appropriate subgraph of H_0 where each component has at most one cutvertex. Assume without loss of generality that $|A_1| \geq |A_2|$. Hence $|A_1| \geq 4q$. Now the wheels whose centers are in A_1 can play the role of H, except that there may be edges joining distinct blocks in the same connected component, say Q, of that subgraph. In other words, Q consists of say m wheels having a cutvertex u in common. In $Q - u$ we contract each component to a single vertex and apply the 4-color theorem to the resulting graph. The largest color class has at least $m/4$ vertices and corresponds to a collection of wheels with no edges between them. Now we have at least $|A_1|/4 \geq q$ wheels that together can play the role of H. This completes the proof of Theorem 1.

We shall apply Theorem 1 to the number of colorings. First we investigate colorings of triangulations with many vertices of degree < 6 contained in rails.

For that we need the following well-known observation.

Lemma 2 *Let G be a connected planar graph with outer cycle C of length 3. Assume that each edge e of G has nonnegative weight $w(e)$. Then G has a spanning tree T containing two edges of C such that $w(T) \geq w(G)/3$.*

Proof of Lemma 2.

As a referee points out, Lemma 2 follows from the existence of so-called Schnyder woods by first triangulating the graph by adding edges of weight zero, and then partitioning the edges inside C into three trees. When the new edges are deleted one of the resulting forests can be extended to a spanning tree of weight $w(T) \geq w(G)/3$.

We present also a short proof from first principles. If G has a separating triangle C', then we apply induction first to C' and its exterior, and then to C' and its interior. In the latter case the weights of C' are considered to be 0. If G has no separating triangle, we let e be an edge inside C of maximum weight, and we contract e and use induction. Before we use induction, we delete an edge in each double edge. There may be two such edges. After the induction we add e to the spanning tree.

Lemma 3 *Let G be a planar near-triangulation with a precolored outer cycle C of length 3 or 4 and with n vertices in the interior. If C has a chord, then the ends have distinct colors.*

(a) If C has length 3, then the coloring of C can be extended to a 5-coloring of G in at least 2^n ways.

(b) If C has length 4, then the coloring of C can be extended to a 5-coloring of G in at least $3 \cdot 2^{n-3}$ ways.

Proof of Lemma 3. The proof is by induction on n. For $n = 1$ the statement is trivial, so we proceed to the induction step.

We may assume that C has no chord. For otherwise, that chord divides G into two parts and we apply induction to each part.

We may also assume that G has no separating triangle, since otherwise, we apply induction first to its exterior and then to its interior.

Then the interior of C has a vertex v which in G has degree at most 5. Now apply the equations (5), (6) and the induction hypothesis. As mentioned at the end of Section 2, we may apply the versions of (5), (6) where some vertices are precolored.

If C has length 3, then four polynomials (counted with multiplicity, that is, a polynomial is counted twice if it is multiplied by $k-3=2$) on the right hand side of (5), (6) are distinct from the zero-polynomial and we complete the proof by induction.

So assume that C has length 4. Let $C : x_1x_2x_3x_4x_1$. Consider first the case where the interior of C has a vertex v which in G has degree at most 5 and which is not contained in a path of length 2 joining non-neighboring vertices on C of distinct color or a path of length 3 joining two non-neighboring vertices of C with the same color. Then we complete the proof as in the case where C has length 3 by applying (5), (6) where some vertices are precolored. This works because none of the polynomials on the right hand side in (5), (6) are the zero-polynomials. So assume that every vertex v inside C which in G has degree at most 5 is contained in a path of length 2 or 3 joining two opposite vertices of C such that we cannot apply induction.

Consider first the case where some vertex v inside C of degree < 6 is joined to x_2, x_4 and these two vertices have distinct color. If all vertices inside C of degree < 6 are joined to x_2, x_4, then all vertices inside C have degree 4. Moreover, together with x_1, x_3 they induce a path, and it is easy to color the vertices one by one such that we have two color choices each time, except for the last vertex where we may have only one choice. Thus the number of color extensions is at least 2^{n-1}. So assume that not all vertices inside C of degree < 6 are joined to x_2, x_4. Then each vertex v' distinct from v inside C of degree < 6 is contained in a path of length 3 joining x_1, x_3, and these two vertices x_1, x_3 have the same color. Then every vertex inside C which is distinct from v and has degree < 6 is joined to v, x_3. Then in fact every vertex inside C which is distinct from v is joined to v, x_3. Together with x_2, x_4 these vertices induce a path. We now color v in two possible ways, and each such coloring can be extended in at least 2^{n-2} ways by a previous argument.

Consider next the case where no vertex v inside C of degree < 6 is joined to two opposite vertices of C with distinct colors. Then every vertex v inside C of degree < 6 is joined to two opposite vertices of C with the same color by a path of length 3.

Let G' be the union of all paths of length 3 joining opposite vertices of C with the same color and containing a vertex of degree < 6. Assume first that these two vertices on C are always x_2, x_4. We can denote these paths in G' by $x_2 v_{1,i} v_{2,i} x_4$ where $x_1, v_{1,1}, v_{1,2}, \ldots$ are neighbors of x_2 in clockwise order. Some of the vertices $x_1, v_{1,1}, v_{1,2}, \ldots$ may be identical, and some of the vertices $x_1, v_{2,1}, v_{2,2}, \ldots$ may be identical. Consider a facial cycle C' of G' which has a vertex of G in its interior. The interior of C' contains no vertex of degree at most 5 by the definition of G'. It is now an easy consequence of Euler's formula, that C' must have length 6 and the interior of the 6-cycle C' contains precisely one vertex, and this vertex has degree 6. So, G' contains all vertices of G, except possibly some of degree 6. It follows that some neighbor of x_1, say $v_{1,1}$, has degree 4. If the interior of C has only the two vertices $v_{1,1}, v_{2,1}$, the proof is easy to complete. So assume that $v_{1,1}$ has a neighbor x_1' inside C and distinct from $v_{2,1}$. By deleting $v_{1,1}$ and adding the edge $x_1 x_1'$ we obtain $3 \cdot 2^{n-4}$ colorings each of which can be extended to G. By deleting $v_{1,1}$ and identifying x_1, x_1' we obtain $3 \cdot 2^{n-5}$ colorings each of which can be extended to G in at least two distinct ways.

Consider finally the case where x_1, x_3 have the same color, x_2, x_4 have the same color, there is a vertex x of degree < 6 and a path $x_2 x z x_4$, and there is a vertex y of degree < 6 and a path $x_1 y z x_3$. Suppose further that there is no path of length 3 joining x_1, x_3 containing x and no path of length 3 joining x_2, x_4 containing y. Again, we let G' be the union of all paths of length 3 containing a vertex of degree < 6 and joining opposite vertices of C. It is now easy to see that G' contains all vertices of G and that $G - x_1 - x_2 - z$ is a cycle $x_3 u_1 u_2 \ldots u_k x_4 x_3$ whose vertices are all joined to z, and precisely one vertex, say u_j, in the cycle is joined to both of x_1, x_2. Now z, u_j can be colored in 6 ways, and there two color choices for all the remaining vertices except possibly two.

This completes the proof of Lemma 3.

For the next result we need some more notation. We consider a planar triangulation G with with outer 3-cycle C. Let S denote the set of vertices in G inside C of degree < 6. We are going to find a lower bound for the number of proper 5-colorings with colors $1, 2, 3, 4, 5$. We assume that C is precolored.

We define an m-rail in G to be *clean* if $m \geq 13$, the two vertices of degree m are non-neighbors in G, all vertices, except the two of degree m are in S, and the rail is maximal with these properties. (In other words, if v is a vertex in S, and v is joined to the two vertices of degree m in the rail, then v is also in the rail.) The $m-2$ vertices in the rail which are not on the outer 4-cycle of the rail are called *inner vertices* of the rail. The vertices of G which are inside the outer 4-cycle of the rail are called *interior vertices* of the rail.

Note that two clean rails may intersect. If so, they have a vertex or an edge or a path of length 2 in common. That vertex or edge or path is contained in the outer 4-cycle of one of the rails. In particular, an inner vertex in one clean rail cannot be an inner vertex in another clean rail. Let q denote the total number of inner vertices of the clean rails, We now define the *rail graph* $R(G)$ with respect to G. The vertices of $R(G)$ are the vertices in the clean rails of degree at least 13. Two vertices u, v in $R(G)$ are neighbors if G has a clean rail, say an m-rail, containing u, v. We assign the weight $m-2$ to the edge uv in $R(G)$. In each component of $R(G)$ we select a spanning tree of maximum weight. By Lemma 2, the weight of any such tree is at least $1/3$ of the total weight of the component because $R(G)$ is planar. By Lemma 1, each such tree has a 2-edge-coloring such that each color class is a forest consisting of stars. We select the color class with maximum weight. Consider now a component in that forest. That component is a star. The vertices in this star are also vertices in G. Using the 4-color theorem we can 4-color the vertices in this star distinct from the center (such that neighbors in G get distinct colors). We select one of the color classes such that the sum of weights from the color class to the center of the star is maximum. We now consider the corresponding clean rails in G. We call these the *very clean rails*. The discussion above shows that the total number of inner vertices in the very clean rails is at least $q/24$.

Theorem 2 *Let q be a natural number, and let G be a planar triangulation with a precolored outer cycle C of length 3 and with n vertices in the interior. Assume also that G has q vertices each of which is an inner vertex in a clean rail.*

Then the coloring of C can be extended to a 5-coloring of G in at least $(5/4)^{q/24} \cdot 2^n$ ways.

Proof of Theorem 2.

Let q_1 be the largest total number of inner vertices in a collection of very clean rails. We prove, by induction on $n + q + q_1$, that the coloring of C can be extended to a 5-coloring of G in at least $(5/4)^{q_1} \cdot 2^n$ ways. This will prove Theorem 2 since the discussion before Theorem 2 shows that $q_1 \geq q/24$.

If $n < 13$, then $q_1 = q = 0$, and the result follows from Lemma 3. So assume $n \geq q \geq q_1 > 0$.

Let q' be the number of inner vertices, and let n' be the number of interior vertices, in a very clean rail R' such that none of the interior vertices of R' are in other very clean rails. Let u, v be the vertices of degree $q' + 2$ in R'. Thus R' has a minimality property. For convenience, we shall choose R' such that it has a stronger minimality property: First we extend R' to a maximal rail R'' (which is not necessarily clean) by adding all common neighbors of u, v. We choose the very clean rail R' such that R'' has no other very clean rail inside the outer 4-cycle of R''.

We delete all interior vertices of R'. Then we identify u, v and call the resulting graph G_1. The number of vertices G_1 inside C is $n - n' - 1$. Note that the cleans rails distinct from R' in our collection of very clean rails are still very clean. So the new q_1 is at least $q_1 - q'$. When we identify u, v we may create separating triangles. This is no problem as we allow separating triangles. We may also create separating cycles of length 2. They are also easy to dispose of by induction since their interiors do not contain any very clean rail, by the minimality property of R'.

So, by induction, the coloring of C can be extended to a 5-coloring of G_1 in at least $(5/4)^{(q_1 - q')} \cdot 2^{n-n'-1}$ ways.

Let G_2 be the graph induced by R' and all its interior vertices. Consider one of the above 5-colorings of G_1. We shall extend it to G_2 in many distinct ways. Let the vertices joining u, v in R' be $x_1, x_2, \ldots, x_{q'+2}$ numbered consecutively. We first extend the coloring of G_1 to the 4-cycle $x_1 u x_2 v x_1$ and its interior, then to $x_2 u x_3 v x_2$ and its interior, and finally to $x_{q'+1} u x_{q'+2} v x_{q'+1}$ and its interior. Each time we use Lemma 3(a) (as explained below), except that we apply Lemma 3(b) when we color the interior of $x_{q'+1} u x_{q'+2} v x_{q'+1}$ because all vertices of the 4-cycle are precolored. In that coloring we loose the factor 3/8. But in the remaining q' cases we gain a factor 3/2 because we identify u, v before we apply Lemma 3(a) as follows: Let us again consider the 4-cycle $x_1 u x_2 v x_1$ and its interior. Then u, v are colored 1, say, and x_1 is colored 2, say. x_2 is not yet colored. Then we identify the vertices u, v such that two cycles of the resulting graph are 2-gons. We delete one edge of each 2-gon to get a triangulation whose outer cycle has two colored vertices. We give the third vertex any color distinct from $1, 2$. Since we have 3 choices (rather than 2) for the third vertex of the outer triangle, we gain the factor $3/2$. So the number of color extensions of the coloring of G_1 is at least $(3/8) \cdot (3/2)^{q'} \cdot 2^{n'}$. So the number of color extensions of C is at least

$(5/4)^{(q_1-q')} \cdot 2^{n-n'-1} \cdot (3/8) \cdot (3/2)^{q'} \cdot 2^{n'}$ which is $> (5/4)^{q_1} \cdot 2^n$ since $q' > 10$.

This completes the proof of Theorem 2.

Theorem 3 *Let T be a planar triangulation with n vertices, and let q be a natural number, $q \geq 6$. Assume that at least $4029q$ vertices of G have degree < 6. Assume that G has no separating triangle. Then*

$$P(G, 5) \geq (5/4)^q \cdot 60 \cdot 2^{n-3}.$$

Proof of Theorem 3. If G has $29q$ vertices which have degree ≤ 5 and which are part of m-rails where $m \geq 13$, then at least $24q$ of these are inner vertices in those rails and we can apply Theorem 2. So assume that G has no such $29q$ vertices.

Then G has $4000q$ vertices satisfying the assumption of Theorem 1. Let H be the graph in the conclusion of Theorem 1. Then we successively apply the inequality (9) q times, and each time we gain the factor $5/4$. After having applied that inequality some times, the resulting graph may have separating triangles. But all we need is that the vertices of degree < 6 we use in the recursion are not contained in separating triangles. We may also create separating cycles of length 2. Again, they are easily disposed of by induction.

4 Infinite 6-regular planar triangulations

In this section the graphs may be infinite. We shall present a complete characterization of the infinite 6-regular triangulations. We shall apply that characterization (or, more precisely, its proof) to finite planar graphs, in particular colorings. In the last section we point out that the characterization can also be used to translate the isoperimetric inequality for the plane into a discrete isoperimetric inequality.

Let T_6 denote the infinite 6-regular triangulation of the plane which is the dual graph of the hexagonal tiling. We can also obtain T_6 from a vertex x_0 by adding concentric cycles $C_1, C_2, \ldots$ of lengths $6, 12, 18, \ldots$ and edges between C_i, C_{i+1}, $i = 0, 1, \ldots$, where $C_0 = x_0$.

Other 6-regular triangulations can be obtained from a quadrangulation (of the plane) which is the cartesian product of a cycle of length m and a two-way infinite path by adding edges in the quadrangles. This quadrangulation has subgraphs which is the cartesian product of a cycle of length m and a K_2. This subgraph has m (or $m+2$ if $m=4$) cycles of length 4. We can add m chords in these m 4-cycles in 2^m ways. For each of these 2^m choices, there is a unique way to add edges inside the other facial 4-cycles in order to obtain a 6-regular triangulation, except if the chords are added such that the two consecutive m-cycles form a 4-regular graph. In that case there are two choices for the chords between the next two consecutive m-cycles. However these two choices lead to isomorphic triangulations. Also, some of the other 2^m triangulations are isomorphic. We call any such triangulation a *6-regular cylinder of circumference m*.

A subgraph of the 6-regular cylinder of circumference m obtained from the cartesian product of a cycle of length m and a path of length q by adding edges in the quadrangles is called a *6-regular cylinder of circumference m and length q*. (Here the term 6-regular is slightly misleading as the vertices on the two boundary cycles of the cylinder have degree < 6.)

Euler's formula implies that if a finite planar graph has a cycle of length m such that all vertices inside C have degree at least 6, then there are at most $2m-6$ edges from C to its interior. In [9] this was used to give a short proof of the following.

Theorem 4 *Let T be an infinite 6-regular triangulation of the plane such that the interior of every cycle is finite. Let y_0 be a vertex in T and let $D_1, D_2, \ldots$ be the distance classes from y_0. Then there is an isomorphism of T_6 onto T taking C_i onto D_i for each $i \geq 1$.*

In [9] Theorem 4 was used for the number of k-colorings of graphs on a fixed surface. We shall here apply it to coloring planar graphs with no separating triangles. We also need the following.

Theorem 5 *Let T be an infinite 6-regular triangulation of the plane. Then T is isomorphic to T_6 or to a 6-regular cylinder.*

Proof of Theorem 5. By Theorem 4 we may assume that T has a cycle C_0 such that both the interior and exterior of C_0 are infinite. Let m denote the length of C_0 and assume that m is chosen to be minimum. Clearly, C_0 is induced (that is, without chords).

Consider now a vertex v outside C_0 which is a neighbor of C_0. We claim that v is joined to one or two or three consecutive vertices of C_0. This claim is trivial if C_0 has length 3. The claim is also easy to prove when C_0 has length at least 5. For if v is outside C_0 and is joined to two nonconsecutive vertices u, w, then there are two cycles containing the path uvw and a path of C_0. One of these contradicts the minimality of C_0 unless one of the two cycles has length 4 and has finite interior (or exterior). But that is not possible because Euler's formula easily implies that there exist no (finite) planar near-triangulation whose outer cycle has length 4 and all other vertices have degree precisely 6 (unless the interior (or exterior) consists of just one edge). So we only need to consider the case where C_0 is a 4-cycle $v_1v_2v_3v_4$, and v is joined to v_1, v_3. Also, v is joined to none of v_2, v_4 (since otherwise v does not contradict the claim), and v_2, v_4 are not neighbors because C_0 is chordless. Now v_1 (respectively v_3) has a neighbor v_1' (respectively v_3') inside the cycle $v_1v_2v_3vv_1$. Because there are similar neighbors inside the two other 4-cycles and v_1, v_3 have degree 6, this accounts for all neighbors of v_1, v_3. Hence v_1', v_3' are joined to v, v_2. If $v_1' = v_3'$, then some triangle containing v_1' is separating. By Euler's formula, such a triangle must have infinitely many vertices in its interior and in its exterior. This contradicts the assumption that C_0 has length 4. So, $v_1' \neq v_3'$. One of v_2, v_4 (say v_2) has at least two neighbors inside C_0 since otherwise, there is a vertex z inside C_0 joined to all vertices of C_0 which results in a separating triangle. Then v_2 has no neighbor outside C_0. Then v_1', v_3' must be neighbors and again, there is a separating triangle, and we obtain a contradiction which proves the claim.

Assume now that C_0 has length at least 4, that is, T has no separating triangle. As T is a triangulation, and a vertex outside C_0 is joined only to consecutive vertices of C_0, it follows that the neighbors of C_0 outside C_0 induce a cycle C_1 of length m_1 say. Similarly, the neighbors of C_0 inside C_0 induce a cycle C_{-1} of length m_{-1}, say. As the number of edges leaving C_0 is $4m$, we have $m_1 + m_{-1} = 2m$. The minimality of m implies that $m_1 = m_{-1} = m$. We repeat this argument defining cycles $C_2, C_3, \dots$ and $C_{-2}, C_{-3}, \dots$.

If each vertex of C_i has two neighbors in C_{i+1} and two neighbors in C_{i-1} for each integer i, then clearly T is isomorphic to the triangulation obtained from the cartesian product of a cycle of length m and a two-way infinite path by adding a perfect matching in the quadrangles. So assume that some vertex v_0 in C_0 has only one neighbor v_1 in C_1. Then v_1 has three neighbors in C_0 and hence only one neighbor v_2 in C_2. Also, there is a vertex v_{-1} in C_{-1}

joined to only v_1 in C_0. We call the two-way infinite path $\ldots v_{-1}v_0v_1\ldots$ a *positive principal path*. A *negative principal path* is defined analogously. The number of positive principal paths equals the number of negative principal paths. As all vertices in C_i between consecutive principal paths have two neighbors in C_{i+1} and two neighbors in C_{i-1}, it is now easy to see that T is a 6-regular cylinder.

There remains only the case that C_0 has length 3. As we shall not use this case in the applications, we only sketch the proof and leave the details for the reader. We first argue that there are at least 6 edges from C_0 to its interior and at least 6 edges from C_0 to its exterior. As there are precisely 12 edges leaving C_0, there are precisely 6 edges from C_0 to its interior and precisely 6 edges from C_0 to its exterior. Now it is easy to see that the neighbors of C_0 inside C_0 induce a triangle, and we can repeat the proof in the case where C_0 has length > 3.

5 Planar triangulations with few vertices of degree < 6

Proposition 1 *Let p, q be natural numbers. Let C be an induced cycle of length p in a planar triangulation with no separating triangle. Assume that both the interior of C and the exterior of C contain a vertex of degree < 6. Assume that all vertices of distance $\leq pq$ from C have degree precisely 6. Then G contains a 6-regular cylinder of length q and circumference at most p.*

Proof of Proposition 1. Let the notation be chosen such that there are at most $2p$ edges from C to its interior. Then the neighbors of C in the interior of C induce a subgraph which contains a cycle C_1 of length $\leq p$. If C_1 can be chosen such that it has length $< p$ and there is a vertex of degree < 6 in its interior, then we consider C_1 instead of C with $|E(C_1)|$ instead of p. So assume that C_1 cannot be chosen in that way. Then we claim that C_1 has length p, C_1 is the graph induced by the neighbors of C inside C, and there are precisely $2p$ edges between C and C_1. (To prove this claim we first observe that there is a walk of length at most p through the neighbors of C_0 inside C_0. The assumption on C_1 implies that that walk cannot have vertex repetitions.) We repeat the argument with C_1 instead of C defining distinct cycles $C, C_1, C_2, \ldots, C_q$ of non-increasing length. If they all have length p, the proof is complete. The proof of Theorem 5 shows that the cycles $C, C_1, C_2, \ldots, C_q$ induce a graph which is a cylinder of length q. If one of them has length $< p$ we consider that cycle instead of C.

Theorem 6 *Let q be a natural number, and let G be a triangulation of the plane with no separating triangle. Let v be a vertex such that all vertices of G of distance $\leq 3q^2 + q$ to v have degree precisely 6. Then either the vertices of distance at most q from v induce a near-triangulation isomorphic to a subgraph in T_6 or else G contains a 6-regular cylinder of length q and circumference at most $3q$.*

Proof of Theorem 6. Let d be the largest natural number such that the vertices of distance at most $d-1$ from v induce a near-triangulation isomorphic to a subgraph in T_6. If $d \geq q+1$, then the vertices of distance at most q from v induce a near-triangulation isomorphic to a subgraph in T_6. So assume that $d \leq q$.

Let $C_1, C_2, \ldots, C_d$ be the subgraphs induced by the vertices of distance $1, 2, \ldots, d$, respectively, from v. Then $C_1, C_2, \ldots, C_{d-1}$ are cycles, whereas C_d is a walk $v_1 v_2 \ldots v_{6d} v_1$ possibly with some chords. The maximality of d implies that C_d is not an induced cycle. Hence we may assume that there is a natural number j distinct from $3, 2, 1, 6d, 6d-1$ such that either $v_1 = v_j$, or there is an edge $v_1 v_j$. We claim that the latter holds. To prove this claim, let us assume that $v_1 = v_j$. Then also j is distinct from $4, 6d-2$ because v_2, v_{6d} have degree 6, and G has no separating triangle. Now the two occurrences v_1, v_j of v_1 divide C_d into two subwalks. We claim that either of them have a chord such that this chord together with a subwalk of C_d form a cycle. To prove this claim, it suffices to consider the case where $v_1 = v_j$, and the walk $v_1 v_2 \ldots v_j$ is a cycle. As v_1, v_j are incident with 6 edges in the subgraph we have already defined, it follows that C_d has the chord $v_2 v_{j-1}$, hence the claim. So we may assume that $v_1 v_j$ is a chord. By letting j be minimal we may assume that G contains the cycle $C'_d : v_1 v_2 \ldots v_j v_1$. By focussing on the walk $v_j v_{j+1} \ldots v_1$ (rather than the path $v_1 v_2 \ldots v_j$) we conclude that there is another similar cycle C''_d having at most one edge in common with C'_d.

We assume that C_{d-1}, C'_d are drawn such that their interiors are pairwise disjoint. Now focus on C'_d. We redraw C'_d on the equator of a sphere. We draw the interior of C'_d on the upper hemisphere. Then we draw the same graph on the lower hemisphere. Let n denote the number of vertices of this graph H. Then all vertices on the equator, except possibly two, have degree at least 6 in H (because each of the vertices $v_2, v_3, \dots, v_{j-1}$ has at most two neighbors in C_{d-1}). If all vertices not on the equator have degree at least 6, then H has more than $3n-6$ edges, a contradiction to Euler's formula. So the interior of C'_d has a vertex of degree < 6. Applying the same argument to C''_d shows that also the exterior of C'_d has a vertex of degree < 6. Now C'_d satisfies the assumption of Proposition 1 with $p = 3q$.

6 The number of colorings of planar graphs with no separating triangles

Theorem 7 *Let G be a planar graph with n vertices and no separating triangle. Then*

$$P(G,5) \geq (2 + 10^{-12})^n.$$

Proof of Theorem 7.

Suppose (reductio ad absurdum) that G is a counterexample with as few vertices as possible and (subject to this condition) with as many edges as possible.

We leave it to the reader to verify the statement when G is a wheel. Suppose therefore that G is not a wheel. We claim that G is a triangulation. For otherwise, we can add an edge e to G such that $G+e$ is planar and has no separating triangle containing e. This is easy if G is disconnected or has a cutvertex. If G is 2-connected and the outer face, say, is not bounded by a triangle, then we try to add an edge between two nonconsecutive vertices on the outer cycle. For any such two vertices we may assume that the two vertices are joined by an edge (in which case it is easy to add another edge), or there is a vertex inside the outer cycle joined to the two vertices. That vertex inside the outer cycle must be the same for all choices of nonconsecutive vertices (since otherwise, it is easy to add another edge) that is, G is a wheel, a contradiction. By equation (1), $G+e$ is a counterexample with more edges than G, a contradiction.

The smallest planar triangulation distinct from K_4 with no separating triangle is the graph of the octahedron (that is a K_6 minus a perfect matching). It has 13 distinct 5-colorings (extending the precolored triangle) which is greater than the desired number. So assume that $n > 6$.

Consider first the case where the number of vertices of G of degree ≤ 5 is at least $n/2^{24}$. Then we apply Theorem 3 with $q = n/(2^{24} \cdot 4029)$.

$$P(G,5) \geq (5/4)^q \cdot 60 \cdot 2^{n-3} \geq (2 + 10^{-12})^n.$$

Assume next that the number of vertices of G of degree ≤ 5 is m where $m \leq n/2^{24}$.

We claim that G has a vertex u such that all vertices of distance at most 1220 from u have degree precisely 6. To prove this claim, let $v_1, v_2, \ldots, v_r$ be the vertices of degree distinct from 6, and let $d_1, d_2, \ldots, d_r$ be their degrees. As the average degree of G is less than 6, it follows that $r < 3m$. (For, if all m vertices of degree < 6 have degree precisely 4, then $2m$ vertices of degree 7 would give average degree 6. As the average degree is < 6, there are less than $2m$ vertices of degree > 6.) Also,

$$d_1 + d_2 + \ldots + d_r < 6r \leq 18m.$$

Consider the case where all vertices of distance ≤ 1220 to v_i have degree precisely 6, for each $i = 1, 2, \ldots, r$. Then there are at most jd_i vertices of distance j to v_i for $i = 1, 2, \ldots, r$, and $j = 1, 2, \ldots, 1220$. So, the number of vertices of distance at most 1220 to v_i is $< (1221)^2 d_i/2$, $i = 1, 2, \ldots, r$. So, the number of vertices of distance at most 1220 to some v_i ($i = 1, 2, \ldots, r$) is $< (1221)^2 9m < n$.

Consider next the general case where some of the vertices $v_1, v_2, \ldots, v_r$ may not be far apart. Let H_d be the subgraph of G induced by the vertices of distance at most d to some of $v_1, v_2, \ldots, v_r$. A *boundary vertex* of H_d is a vertex joined to a vertex not in H_d. Let a_d, respectively b_d, be the number of boundary vertices of H_d-degree 3, respectively > 3. Consider a face boundary of H_d bounding a face which contains vertices not in H_d. For each edge e of this face boundary, there is a triangle with edges e, e', e'' such that e', e'' are outside H_d. There may be other edges from H_d to the face we focus on, but only from the a_d vertices of degree precisely 3 in H_d. The common end of e', e'' has degree at least 4 in H_{d+1}. These observations imply that $a_{d+1} \leq a_d$ and, for the total number of boundary vertices we have $a_{d+1} + b_{d+1} \leq (a_d + b_d) + a_d$. These two inequalities are equalities for some triangulations where all of $v_1, v_2, \ldots, v_r$ are far apart. Therefore, also in the general case we conclude that the number of vertices of distance at most 1220 to some v_i ($i = 1, 2, \ldots, r$) is $< (1221)^2 9m < n$.

This proves the claim that G has a vertex v such that all vertices of distance at most 1220 from u have degree precisely 6.

Now we apply Theorem 6 with $q = 20$, and we let v be as in Theorem 6, or v is a vertex in the middle of the cylinder, that is, v has distance 10 to the two boundaries of the cylinder.

Consider first the case where the vertices of distance at most 20 from v induce a subgraph of T_6. We delete v and identify three of its neighbors. By induction, the number of 5-color-extensions of the resulting graph and hence also G is at least $(2 + 10^{-12})^{n-3}$. This is too little, and we express this by saying that *we have lost the factor* $(2+10^{-12})^3$. But now there is a neighbor of v which has degree at most 5. Then we use the reduction formulas (5), (6) for that vertex, and thereby *we gain the factor* 5/4. We repeat this, and each time we gain the factor 5/4. To do this it is important that there are no separating triangles. So, if there is a separating triangle, we delete its interior before we use induction. After the induction we put the vertices in the interior back. If the interior has r vertices, then they can be colored in 2^r ways by Lemma 3. So we loose the factor $(1 + 10^{-12}/2)^r$ whenever there is such a separating triangle. We proceed like this and stop when we have gained the factor 5/4 ten times.

We now argue why we can do this at least ten times. Suppose that we have used the reduction formulas (5), (6) p times, where $p < 10$. Our current graph is obtained from G by successively contracting some edges. We also delete the interior of separating triangles whenever they appear (but such an operation may also be thought of as a series of edge-contractions.) We claim that all these operations involve vertices of distance at most $2p$ to v and can be continued if $p < 10$. To prove this claim, we think of the vertices in G of distance at most 20 from v as the vertices in the infinite triangulation T_6 of distance at most 20 from v. Let i be the smallest number such that no vertex in C_i (the cycle in T_6 whose vertices have distance i from v) is part of any of the p operations above. If an edge of C_{i-1} has been contracted, then the corresponding vertex of C_i has gotten degree < 6. So assume that no edge of C_{i-1} has been contracted. Let u_1 be a vertex C_{i-1} that has been identified with a neighbor u_1' in C_{i-2}. Let u_2 be a common neighbor of u_1, u_1' on C_{i-1}. Now u_2 has degree < 6 unless u_2 has been identified with a neighbor u_2' in C_{i-2}. We continue defining $u_3, u_4, \ldots$. But this sequence must terminate when we reach a vertex in C_{i-1} that has only one neighbor in C_{i-2}.

So we have gained the factor $(5/4)^{10} > 9$. We shall now give an upper bound for the factors we loose. At the beginning we lost the factor $(2 +$

$10^{-12})^3$. All reductions take place among vertices that have distance ≤ 20 from v. So the total number of vertices inside separating triangles is less than $6 + 12 + \ldots + 19 \cdot 6 = 1140$. So we lose altogether a factor less than $(2 + 10^{-12})^3(1 + 10^{-12}/2)^{1140}$. As the gain $(5/4)^{10}$ is greater than the loss $(2 + 10^{-12})^3(1 + 10^{-12}/2)^{1140}$, the proof is complete.

If v is in the middle of a cylinder we argue similarly except that now we may encounter a separating triangle which separates the two boundaries of the cylinder. In that case we use that triangle to separate the current graph into three triangulations and we repeat the above proof for two of them, as explained below.

More precisely, if we encounter a separating triangle which separates the two boundaries of the cylinder, then we consider a separating triangle T_1 such that the part G_1 of the graph containing one of the boundary cycles is smallest possible, and we consider a separating triangle T_2 such that the part G_2 of the graph containing the other boundary is smallest possible. We let G_3 consist of T_1, T_2 and all the vertices between them, that is the vertices in neither of G_1, G_2. When we do induction, we color first G_1 by induction, then we color G_3 using Lemma 3, and finally we color G_2 by induction. In order to get good lower bounds we also apply the reduction formulas (5), (6) as often as possible to G_1, G_3 as in the case where the distance classes from v induce a subgraph of T_6. We compare gain and loss as in that part of the proof. In the present part the loss is smaller because the distance classes in the cylinder are smaller than (or of same size) than those in T_6. Now we need only 10 distance classes from v (as opposed to 20 when the distance classes agree with T_6) because we can find a vertex of degree < 6 close to each of the boundary cycles. We need an additional argument to find a vertex of degree < 6 because it is indeed possible to contract finitely many edges in a 6-regular cylinder and obtain a graph isomorphic to the cylinder. However this can only be done by contracting perfect matchings between consecutive cycles in the cylinder. And because of the initial operation where we delete v and identify three neighbors, we cannot get the cylinder itself by edge-contractions, and hence there will be a vertex (in fact two vertices) of degree < 6.

7 Planar graphs with no separating triangles and few 5-colorings

It is now natural to ask how much the constant $2+10^{-12}$ can be increased in Theorem 7. Consider the 6-regular cylinder of length q and circumference 4 such that all vertices on the boundary cycles have degree 4. There is only one such cylinder up to isomorphism. This cylinder has $n = 4q$ vertices. Let C_0 denote one of the two boundary cycles. Let α_q, respectively β_q, respectively γ_q denote the number of colorings of this cylinder in colors $1, 2, 3, 4, 5$ such that the boundary cycle C_0 has precisely 2, respectively 3, respectively 4 distinct colors. Then it is easy to check that, for each natural number $q \geq 2$,

$$\alpha_q = 6\alpha_{q-1} + 2\beta_{q-1},$$

$$\beta_q = 12\alpha_{q-1} + 16\beta_{q-1} + 8\gamma_{q-1},$$
$$\gamma_q = 8\beta_{q-1} + 18\gamma_{q-1}.$$

To verify this, consider all $\alpha_{q-1} + \beta_{q-1} + \gamma_{q-1}$ 5-colorings of the cylinder obtained by deleting C_0. For each of these colorings, we then express the number of color extensions to C_0.

The coefficient matrix has three real eigenvalues, the largest being 25.6.... Hence each of $\alpha_q, \beta_q, \gamma_q$ is bounded above by a constant times $25.7^q = 25.7^{n/4}$. Hence $\alpha_q + \beta_q + \gamma_q < 2.252^n$ for n large.

8 An isoperimetric inequality

In this section we point out that the characterization in Theorem 4 has another application.

The vertices in T_6 which have distance d to a vertex v induce a cycle C_d of length $m = 6d$. The number q of vertices inside C_d is

$$q = 1 + 6 + 12 + \ldots 6(d-1) = 1 + 3d(d-1) = 1 + m(m-6)/12.$$

So,
$$(m-6)^2 < 12q.$$

This inequality shows that the isoperimetric inequality in the next result is close to best possible.

Theorem 8 *Let H be a finite near-triangulation of the plane, that is, H has an outer facial cycle of C, and all other faces are bounded by triangles. Let m be the length of C, and let q be the number of vertices inside C. Assume that all vertices on C have degree ≤ 4 and that all vertices inside C have degree precisely 6. Then H is a subgraph of T_6, and*

$$m^2 > 2\sqrt{3}\pi q > 10.88q.$$

Proof of Theorem 8. We draw H inside the disc of radius 1 centered at the origin. We extend H to an infinite 6-regular triangulation by adding new cycles $C_1, C_2, \ldots$ and new edges between them. We draw all vertices of C_i on the circle of radius i centered at the origin. It suffices to explain how C_1 is constructed. First we add, for each edge e in C a triangle containing e such that the new vertex is of distance 1 from the origin. All the new vertices corresponding to distinct edges of C are distinct. If a vertex v of C has degree < 4 in H, then we add new edges going to vertices on the unit circle such that v gets degree 6. Now the unit circle will be the cycle C_1. Note that all the new vertices have degree 3 or 4 so we can repeat this construction. The resulting triangulation satisfies the assumption of Theorem 4, and hence it is isomorphic to T_6 which can be drawn in the plane such that all edges have length 1 and all facial triangles have area $\sqrt{3}/4$. In this drawing C has length $l(C) = m$. The area of the union of the six triangles containing an interior vertex is $3\sqrt{3}/2$. Hence the interior of C has area $a(C) \geq q\sqrt{3}/2$. By the isoperimetric inequality for the plane, the circle with circumference m has area $m^2/4\pi$ which is greater than the area $a(C)$. This completes the proof.

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